

THE DIRAC OPERATOR ON LOCALLY REDUCIBLE RIEMANNIAN MANIFOLDS

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ABSTRACT. In this note, we get estimates on the eigenvalues of the Dirac operator on locally reducible Riemannian manifolds, in terms of the eigenvalues of the Laplace-Beltrami operator and the scalar curvature. These estimates are sharp, in the sense that, for the first eigenvalue, they reduce to the result [12] of Alexandrov.

1. INTRODUCTION

We suppose that (M^n, g) is a closed Riemannian manifold with a fixed spin structure. We understand the spin structure as a reduction $SpinM^n$ of the $SO(n)$ -principal bundle of M^n to the universal covering $Ad : Spin(n) \rightarrow SO(n)$ of the special orthogonal group. The spinor bundle $\Sigma M^n = SpinM^n \times_{\rho} \Sigma_n$ on M^n is the associated complex $2^{\lfloor \frac{n}{2} \rfloor}$ dimensional complex vector bundle, where ρ is the complex spinor representation. The tangent bundle TM^n can be regarded as $TM^n = SpinM^n \times_{Ad} \mathbb{R}^n$. Consequently, the Clifford multiplication on ΣM^n is the fibrewise action given by

$$\begin{aligned} \mu : TM^n \otimes \Sigma M^n &\longrightarrow \Sigma M^n \\ X \otimes \psi &\longmapsto X \cdot \psi. \end{aligned}$$

On the spinor bundle ΣM^n , one has a natural Hermitian metric, denoted as the Riemannian metric by $\langle \cdot, \cdot \rangle$. The spinorial connection on the spinor bundle induced by the Levi-Civita connection ∇ on M^n will also be denoted by ∇ . The Hermitian metric $\langle \cdot, \cdot \rangle$ and spinorial connection ∇ are compatible with the Clifford multiplication μ . That is

$$\begin{aligned} X \langle \phi, \varphi \rangle &= \langle \nabla_X \phi, \varphi \rangle + \langle \phi, \nabla_X \varphi \rangle \\ \langle X \cdot \phi, X \cdot \varphi \rangle &= |X|^2 \langle \phi, \varphi \rangle \\ \nabla_X (Y \cdot \phi) &= \nabla_X Y \cdot \phi + Y \cdot \nabla_X \phi, \end{aligned}$$

for $\forall \phi, \varphi \in \Gamma(\Sigma M^n)$ and $\forall X, Y \in \Gamma(TM^n)$. Using a local orthonormal frame field $\{e_1, \dots, e_n\}$, the spinorial connection ∇ , the Dirac operator D and the twistor

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operator P , are locally expressed as

$$\nabla_{e_k}\psi = e_k(\psi) + \frac{1}{4}e_i \cdot \nabla_{e_k}e_i \cdot \psi \quad (1.1)$$

and

$$D\psi \triangleq e_i \cdot \nabla_{e_i}\psi, \quad (1.2)$$

$$P\psi \triangleq e_i \otimes (\nabla_{e_i}\psi + \frac{1}{n}e_i \cdot D\psi) \quad (1.3)$$

which satisfy the following important relation

$$|\nabla\psi|^2 = |P\psi|^2 + \frac{1}{n}|D\psi|^2,$$

for any $\psi \in \Gamma(\Sigma M^n)$. (Throughout this paper, the Einstein summation notation is always adopted.)

Let $R_{X,Y}Z \triangleq (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z$ be the Riemannian curvature of (M^n, g) and denote by $\mathcal{R}_{X,Y}\psi \triangleq (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})\psi$ the spin curvature in the spinor bundle ΣM^n . They are related via the formula

$$\mathcal{R}_{X,Y}\psi = \frac{1}{4}g(R_{X,Y}e_i, e_j)e_i \cdot e_j \cdot \psi. \quad (1.4)$$

We also use the notation

$$R_{ijkl} \triangleq g(R_{e_i, e_j}e_k, e_l)$$

and $R_{ij} = \langle Ric(e_i), e_j \rangle \triangleq R_{ikkkj}$, $Scal = R_{ii}$. With the help of the Bianchi identity, (1.4) implies

$$e_i \cdot \mathcal{R}_{e_j, e_i}\psi = -\frac{1}{2}Ric(e_j) \cdot \psi, \quad (1.5)$$

which in turn gives $2e_i \cdot e_j \cdot \mathcal{R}_{e_i, e_j}\psi = Scal \psi$. Hence one derives the well-known Schrödinger-Lichnerowicz formula

$$D^2 = \nabla^*\nabla + \frac{1}{4}ScalId, \quad (1.6)$$

where ∇^* is the formal adjoint of ∇ with respect to the natural Hermitian inner product on ΣM^n . The formula shows the close relation between $Scal$ and the Dirac operator D .

From (1.6), it follows easily that if λ is an eigenvalue of D , then

$$\lambda^2 \geq \frac{1}{4}Scal_{\min},$$

where $Scal_{\min} \triangleq \min_M Scal$. Clearly, this inequality is interesting only for manifolds with positive scalar curvature, but the minimal value $\frac{1}{4}Scal_{\min}$ cannot be achieved for such manifolds.

The problem of finding optimal lower bounds for the eigenvalues of the Dirac operator on closed manifolds was for the first time considered in 1980 by Friedrich. Using the Lichnerowicz formula and a modified spin connection, he proved the following sharp inequality:

$$\lambda^2 \geq c_n \text{Scal}_{\min}, \quad (1.7)$$

where $c_n = \frac{n}{4(n-1)}$. The case of equality in (1.7) occurs iff (M^n, g) admits a non-trivial spinor field ψ called a real *Killing spinor*, satisfying the following overdetermined elliptic equation

$$\nabla_X \psi = -\frac{\lambda}{n} X \cdot \psi, \quad (1.8)$$

where $X \in \Gamma(TM)$ and the dot “ \cdot ” indicates the Clifford multiplication. The manifold must be a locally irreducible Einstein manifold.

The dimension dependent coefficient $c_n = \frac{n}{4(n-1)}$ in the estimate can be improved if one imposes geometric assumptions on the metric. Kirchberg [8] showed that for Kähler metrics c_n can be replaced by $\frac{n+2}{4n}$ if the complex dimension $\frac{n}{2}$ is odd, and by $\frac{n}{4(n-2)}$ if $\frac{n}{2}$ is even. Alexandrov, Grantcharov, and Ivanov [10] showed that if there exists a parallel one form on M^n , then c_n can be replaced by c_{n-1} . Later, Moroianu and Ornea [13] weakened the assumption on the 1-form from parallel to harmonic with constant length. Note the condition that the norm of the 1-form being constant is essential, in the sense that the topological constraint alone (the existence of a non-trivial harmonic 1-form) does not allow any improvement of Friedrich’s inequality (see [5]). The generalization of [10] to locally reducible Riemannian manifolds was achieved by Alexandrov [12], extending earlier work by Kim [9].

Another natural way to study the Dirac eigenvalues consists in comparing them with those of other geometric operators. Hijazi’s inequality is already of that kind. As for spectral comparison results between the Dirac operator D and the scalar Laplace operator Δ , the first ones were proved by Bordoni [1]. They rely on a very nice general comparison principle between two operators satisfying some kind of Kato-type inequality. Bordoni’s results were generalized by Bordoni and Hijazi in the Kähler setting [2].

In this note, we find a new method which can recover the result of [10] and [13], moreover enable us to use the general spectral result of Bordoni, to show the following theorem

Theorem 1. *Suppose there exists a non-trivial parallel one form on an n -dimensional closed Riemannian spin manifold (M^n, g) , $n \geq 3$ and $\text{Scal} \geq 0$. Let $\lambda_\alpha := \lambda_\alpha(D)$ and consider the first N nonnegative eigenvalues, $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$, then for any positive integer N , we have*

$$\lambda_N(D)^2 \geq \frac{n-1}{n} c \lambda_{k+1}(\Delta) + \frac{n-1}{4(n-2)} \text{Scal}_{\min},$$

where $k = \left\lceil \frac{N}{2^{\lfloor \frac{n}{2} \rfloor + 1}} \right\rceil$, $c = \frac{1}{8(2^{\lfloor \frac{n}{2} \rfloor + 1})^2}$.

In the last section, using the same argument we also obtain similar estimate on locally reducible Riemannian spin manifolds, which generalizes the result of [12] to arbitray eigenvalue $\lambda_N(D)$.

Theorem 2. *Let M be a locally reducible Riemannian spin compact manifolds with positive scalar. Suppose $TM = T_1 \oplus \cdots \oplus T_k$, where T_i are parallel distributions of dimension $n_i, i = 1, \dots, k$, and $n_1 > n_2 \geq \cdots \geq n_k$. Then for any positive integer N ,*

$$\lambda_N(D)^2 \geq \frac{1}{1 + 2 \sum_{i=2}^k \varepsilon_i^{-1}} \frac{n_1}{n_1 - 1} c \lambda_{k+1}(\Delta) + \frac{n_1}{4(n_1 - 1)} \text{Scal}_{\min},$$

where $k = \left\lceil \frac{N}{2^{\lfloor \frac{n}{2} \rfloor + 1}} \right\rceil$, $c = \frac{1}{8(2^{\lfloor \frac{n}{2} \rfloor + 1})^2}$, $\varepsilon_i \triangleq \frac{n_1}{n_i} - 1 > 0$, $i = 2, \dots, k$.

2. THE J -TWIST D_J OF THE DIRAC OPERATOR

Let (M^n, g) be an oriented n -Riemannian manifold. Let J be a $(1, 1)$ -tensor field on (M^n, g) such that $J^2 = \sigma Id, \sigma = \pm 1$ and

$$g(J(X), J(Y)) = g(X, Y),$$

for all vector fields $X, Y \in \Gamma(TM^n)$ (Here Id stands for the identity map). We say (M^n, g, J) is an *almost Hermitian manifold* if $\sigma = -1$ and an *almost product Riemannian manifold* if $\sigma = 1$, respectively. Moreover if $\sigma = -1$ and J is parallel, (M^n, g, J) is called a Kähler manifold. Similarly, we have the following definition.

Definition 1. [11, 9] *An n -Riemannian manifold (M^n, g) is called locally decomposable if it is an almost product Riemannian manifold (M^n, g, J) and J is parallel.*

Example 1. *Suppose an n -Riemannian manifold (M^n, g) possessing a unit vector field $\xi \in \Gamma(TM^n)$, then the reflection J defined by*

$$J(X) \triangleq X - 2g(X, \xi)\xi, \quad X \in \Gamma(TM)$$

is an almost product Riemannian structure. Moreover, it is a locally decomposable Riemannian structure if ξ is a parallel vector field.

As in almost Hermitian spin manifolds, we can also define on almost product Riemannian spin manifolds the following J -twist D_J of the Dirac operator D by

$$D_J \psi \triangleq e_i \cdot \nabla_{J(e_i)} \psi = J(e_i) \cdot \nabla_{e_i} \psi. \quad (2.1)$$

It is not difficult to check that D_J is a formally self-adjoint elliptic operator with respect to L^2 -product, if M^n is closed and $\text{div} J \triangleq (\nabla_{e_i} J)(e_i) = 0$. In fact, let we define a complex vector field

$$\eta(X) \triangleq \langle \phi, J(X) \cdot \psi \rangle = g(Y_1, X) + \sqrt{-1}g(Y_2, X),$$

then

$$\operatorname{div}(Y_1) + \sqrt{-1} \operatorname{div}(Y_2) = e_i(\eta(e_i)) - \eta(\nabla_i e_i) = -\langle D_J \phi, \psi \rangle + \langle \phi, D_J \psi \rangle + \langle \phi, \operatorname{div}(J) \cdot \psi \rangle.$$

Hence the spectrum of D_J is discrete and real.

As in the Kählerian case, Kim obtained the following useful lemma

Lemma 1. $D^2 = D_J^2$ holds on any locally decomposable Riemannian spin manifold (M^n, g, J) (For the general case, see Prop. 2.1 in [9]).

Proof. From now on, we do all calculations at a point $p \in M^n$ where we have an orthonormal frame $\{e_i\}$ which satisfies $(\nabla e_i)|_p = 0, 1 \leq i \leq n$. Hence, for any spinor field ϕ , one has

$$\begin{aligned} D_J^2 \phi &= e_i \cdot \nabla_{J(e_i)}(e_j \cdot \nabla_{J(e_j)} \phi) \\ &= e_i \cdot e_j \cdot \nabla_{J(e_i)} \nabla_{J(e_j)} \phi \\ &= \frac{1}{2} e_i \cdot e_j \cdot \nabla_{J(e_i)} \nabla_{J(e_j)} \phi + \frac{1}{2} e_j \cdot e_i \cdot \nabla_{J(e_j)} \nabla_{J(e_i)} \phi. \end{aligned}$$

Note $e_j \cdot e_i = -e_i \cdot e_j - 2\delta_{ij}$ and $-\nabla_{J(e_i)} \nabla_{J(e_j)} \phi = \nabla^* \nabla \phi$, so we obtain

$$D_J^2 \phi = \frac{1}{2} e_i \cdot e_j \cdot \mathcal{R}_{J(e_i), J(e_j)} \phi + \nabla^* \nabla \phi.$$

But $\nabla J = 0$ and $g(J(X), J(Y)) = g(X, Y)$ imply that

$$\begin{aligned} \mathcal{R}_{J(e_i), J(e_j)} \phi &= \frac{1}{4} g(R_{J(e_i), J(e_j)} e_k, e_l) e_k \cdot e_l \cdot \psi \\ &= \frac{1}{4} g(R_{e_k, e_l} J(e_i), J(e_j)) e_k \cdot e_l \cdot \psi \\ &= \frac{1}{4} g(R_{e_k, e_l} e_i, e_j) e_k \cdot e_l \cdot \psi = \mathcal{R}_{e_i, e_j} \phi, \end{aligned}$$

hence by (1.6), we get $D^2 \phi = D_J^2 \phi$.

In addition, we also need the following lemmas

Lemma 2. Let (M^n, g, J) be a locally decomposable Riemannian spin manifold. If $\psi_\alpha \in E_{\lambda_\alpha}(D) \triangleq \{\psi \neq 0 : D\psi = \lambda_\alpha \psi\}$, then $D_J \psi_\alpha \in E_{\lambda_\alpha}(D) \oplus E_{-\lambda_\alpha}(D)$.

Proof. If $D\psi_\alpha = \lambda_\alpha \psi_\alpha$ and $\lambda_\alpha^2 \neq \lambda_\beta^2$ then Lemma 1 yields

$$\begin{aligned} \lambda_\alpha^2 \int_M \langle \psi_\alpha, D_J \psi_\beta \rangle &= \int_{M^n} \langle D^2 \psi_\alpha, D_J \psi_\beta \rangle \\ &= \int_{M^n} \langle \psi_\alpha, D_J^3 \psi_\beta \rangle \\ &= \int_{M^n} \langle \psi_\alpha, D_J(D^2 \psi_\beta) \rangle \\ &= \lambda_\beta^2 \int_{M^n} \langle \psi_\alpha, D_J \psi_\beta \rangle, \end{aligned}$$

therefore $\int_M \langle D_J \psi_\alpha, \psi_\beta \rangle = \int_M \langle \psi_\alpha, D_J \psi_\beta \rangle = 0$. That is, $D_J \psi_\alpha \in E_{\lambda_\alpha}(D) \oplus E_{-\lambda_\alpha}(D)$. Q.E.D.

Lemma 3. *If ξ is a harmonic vector field of unit length, one gets for any spinor field $\phi \in \Gamma(\Sigma M^n)$ the following Lichnerowicz-type formula*

$$D_J^2 \phi = -\xi \cdot \nabla^* \nabla (\xi \cdot \phi) + \frac{\text{Scal}}{4} \phi, \quad (2.2)$$

where $D_J \triangleq D - 2\xi \cdot \nabla_\xi$. Consequently, we can obtain that

$$\int_{M^n} |D_J \phi|^2 = \int_{M^n} |D(\xi \cdot \phi)|^2. \quad (2.3)$$

Proof. First, $D_J \triangleq D - 2\xi \cdot \nabla_\xi$ is a formally self-adjoint operator since ξ is a harmonic vector field of unit length. Moreover, we can check that for any spinor field ϕ ,

$$\begin{aligned} D_J^2 \phi &= D_J(D\phi - 2\xi \cdot \nabla_\xi \phi) \\ &= D^2 \phi - 2D(\xi \cdot \nabla_\xi \phi) - 2\xi \cdot \nabla_\xi(D\phi) + 4\xi \cdot \nabla_\xi(\xi \cdot \nabla_\xi \phi) \\ &\stackrel{(1.5)}{=} D^2 \phi + \xi \cdot \text{Ric}(\xi) \cdot \phi + 2\xi \cdot e_i \cdot \nabla_{[e_i, \xi]} \phi, \end{aligned}$$

where

$$\begin{aligned} 2\xi \cdot e_i \cdot \nabla_{[e_i, \xi]} \phi &= 2\langle \nabla_i \xi, e_j \rangle \xi \cdot e_i \cdot \nabla_j \phi \\ &= 2\langle \nabla_j \xi, e_i \rangle \xi \cdot e_i \cdot \nabla_j \phi \\ &= 2\xi \cdot \nabla_j \xi \cdot \nabla_j \phi \\ &= \xi \cdot [\nabla_j \nabla_j (\xi \cdot \phi) - (\nabla_j \nabla_j \xi) \cdot \phi - \xi \cdot \nabla_j \nabla_j \phi] \\ &= -\xi \cdot \nabla^* \nabla (\xi \cdot \phi) + \xi \cdot \nabla^* \nabla \xi \cdot \phi - \nabla^* \nabla \phi. \end{aligned}$$

So using the Bochner-Weitzenböck formula on 1-forms

$$\Delta \xi = \nabla^* \nabla \xi + \text{Ric}(\xi) \quad (2.4)$$

one obtains

$$\begin{aligned} D_J^2 \phi &= D^2 \phi - \xi \cdot \nabla^* \nabla (\xi \cdot \phi) - \nabla^* \nabla \phi \\ &= \frac{Scal}{4} \phi - \xi \cdot \nabla^* \nabla (\xi \cdot \phi). \end{aligned} \quad (2.5)$$

Moreover, integrating (2.5) yields,

$$\begin{aligned} \int_{M^n} |D_J \phi|^2 &= \int_{M^n} \langle D_J^2 \phi, \phi \rangle = \int_{M^n} \langle \frac{Scal}{4} \phi - \xi \cdot \nabla^* \nabla (\xi \cdot \phi), \phi \rangle \\ &= \int_{M^n} \frac{Scal}{4} |\phi|^2 + |\nabla (\xi \cdot \phi)|^2 \\ &= \int_{M^n} \frac{Scal}{4} |\xi \cdot \phi|^2 + |\nabla (\xi \cdot \phi)|^2 \\ &= \int_{M^n} |D(\xi \cdot \phi)|^2. \end{aligned}$$

Q.E.D.

Remark 1. If ξ is a parallel one form of unit length, then $D^2 = D_J^2$ by Lemma 1 and hence

$$\int_{M^n} |D\phi|^2 = \int_{M^n} |D_J \phi|^2 = \int_{M^n} |D(\xi \cdot \phi)|^2. \quad (2.6)$$

Furthermore, it also follows that for any spinor field ϕ ,

$$\begin{aligned} \Re \int_{M^n} \langle \xi \cdot \nabla_\xi \phi, D\phi \rangle &= \frac{1}{2} \Re \int_{M^n} \langle (D - D_J)\phi, D\phi \rangle \\ &= \frac{1}{4} \int_{M^n} [|D_J \phi - D\phi|^2 - |D_J \phi|^2 + |D\phi|^2] \\ &\stackrel{(2.6)}{=} \frac{1}{4} \int_{M^n} |D_J \phi - D\phi|^2 = \int_{M^n} |\nabla_\xi \phi|^2. \end{aligned} \quad (2.7)$$

In particular, we conclude that for $\psi_\alpha \in E_{\lambda_\alpha}(D)$

$$\lambda_\alpha \int_{M^n} \langle \xi \cdot \nabla_\xi \psi_\alpha, \psi_\alpha \rangle = \int_{M^n} |\nabla_\xi \psi_\alpha|^2. \quad (2.8)$$

3. MODIFIED CONNECTIONS AND SOME ESTIMATION

We first recall a general spectral comparison result due to Bordoni.(see [1], Theorems 3.2 and 3.3).

Theorem 3. [1] Let (M^n, g) be a closed Riemannian manifold. Let E be any vector bundle of rank p on M , endowed with a Hermitian inner product $\langle \cdot, \cdot \rangle$ and a compatible connection ∇^E . Let \mathcal{R} be any field of symmetric endomorphisms of

the fibers, and define the scalar $R_{\min}(x)$ as the minimal eigenvalue of \mathcal{R}_x acting on E_x . Then, for any positive integer N , we have:

$$\lambda_N(\nabla^{E*}\nabla^E + \mathcal{R}) \geq (1 - c)\lambda_1(\Delta + \mathcal{R}_{\min}) + c\lambda_{k+1}(\Delta + \mathcal{R}_{\min}) \quad (3.1)$$

where

$$k = \left\lfloor \frac{N}{p+1} \right\rfloor, \quad c = \frac{1}{8(p+1)^2}$$

and Δ is the Laplace-Beltrami operator acting on functions.

We shall make use of a modified connection $\nabla^E = \nabla^{(a,b)}$ acting on the spinor bundle $\Gamma(\Sigma M^n)$ of a spin manifold admitting a nontrivial parallel one-form, to which we apply Theorem 2. For any couple a, b of nontrivial real numbers, define the connection $\nabla^{(a,b)}$ by

$$\nabla_X^{(a,b)}\psi \triangleq \nabla_X\psi + aX \cdot \psi + bJ(X) \cdot \psi,$$

where J is the reflection defined by a parallel one-form.

Proposition 1. *The modified connection $\nabla^{(a,b)}$ is compatible with the Hermitian inner product on the spinor bundle. That is, for any vector field X , $\forall \phi, \varphi \in \Gamma(\Sigma M^n)$ one has*

$$X\langle \phi, \varphi \rangle = \langle \nabla_X^{(a,b)}\phi, \varphi \rangle + \langle \phi, \nabla_X^{(a,b)}\varphi \rangle.$$

Proof. A direct computation gives

$$\begin{aligned} \langle \nabla_X^{(a,b)}\phi, \varphi \rangle + \langle \phi, \nabla_X^{(a,b)}\varphi \rangle &= \langle \nabla_X\phi + aX \cdot \phi + bJ(X) \cdot \phi, \varphi \rangle \\ &\quad + \langle \phi, \nabla_X\varphi + aX \cdot \varphi + bJ(X) \cdot \varphi \rangle \\ &= \langle \nabla_X\phi, \varphi \rangle + \langle \phi, \nabla_X\varphi \rangle \\ &= X\langle \phi, \varphi \rangle, \end{aligned}$$

since $\langle X \cdot \phi, \varphi \rangle = -\langle \phi, X \cdot \varphi \rangle$ for any vector field X .

Proposition 2. *The rough laplacian of the modified connection $\nabla^{(a,b)}$ is given by*

$$\nabla^{(a,b)*}\nabla^{(a,b)}\psi = \nabla^*\nabla\psi - 2aD\psi - 2bD_J\psi + [2(n-2)ab + n(a^2 + b^2)]\psi. \quad (3.2)$$

Proof. Consider a local orthonormal frame field $\{e_1, \dots, e_n\}$ and normal coordinates at any fixed point $x \in M$. We have, at this point:

$$\begin{aligned} \nabla^{(a,b)*}\nabla^{(a,b)}\psi &= -\nabla_i^{(a,b)}\nabla_i^{(a,b)}\psi \\ &= -\nabla_i[\nabla_i\psi + ae_i \cdot \psi + bJ(e_i) \cdot \psi] \\ &\quad -ae_i \cdot [\nabla_i\psi + ae_i \cdot \psi + bJ(e_i) \cdot \psi] \\ &\quad -bJ(e_i) \cdot [\nabla_i\psi + ae_i \cdot \psi + bJ(e_i) \cdot \psi] \\ &= \nabla^*\nabla\psi - 2aD\psi - 2bD_J\psi + [2tr(J)ab + n(a^2 + b^2)]\psi \end{aligned}$$

Q.E.D.

Let $\lambda_\alpha \triangleq \lambda_\alpha(D)$ and consider the first N nonnegative eigenvalues, $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$ and $D\psi_\alpha = \lambda_\alpha \psi_\alpha$, $\alpha = 1, \dots, N$. Let $a = b = \frac{\lambda_N}{2(n-1)}$, $J(e_i) \triangleq e_i - 2\langle e_i, \xi \rangle \xi$, here ξ is a harmonic vector field of unit length and write ∇^{λ_N} short for $\nabla^{(\frac{\lambda_N}{2(n-1)}, \frac{\lambda_N}{2(n-1)})}$. In particular, for $N = 1$ we have the following modified connection for $\phi \in \Gamma(\Sigma M^n)$

$$\nabla_X^{\lambda_1} \phi \triangleq \nabla_X \phi + \frac{\lambda_1}{2(n-1)} X \cdot \phi + \frac{\lambda_1}{2(n-1)} J(X) \cdot \phi. \quad (3.3)$$

Then, for the first eigenspinor ψ_1 an elementary calculation provides the following

$$|\nabla^{\lambda_1} \psi_1|^2 = |\nabla \psi_1|^2 + \frac{1}{2(n-1)} (|D_J \psi_1 - D\psi_1|^2 - |D_J \psi_1|^2 - |D\psi_1|^2). \quad (3.4)$$

Observe that

$$|D_J \psi_1 - D\psi_1|^2 = |2\xi \cdot \nabla_\xi \psi_1|^2 = 4|\nabla_\xi \psi_1|^2 = 4|\nabla_\xi^{\lambda_1} \psi_1|^2 \leq 4|\nabla^{\lambda_1} \psi_1|^2 \quad (3.5)$$

and the min-max principle gives

$$\int_{M^n} |D_J \psi_1|^2 \stackrel{(2.3)}{=} \int_{M^n} |D(\xi \cdot \psi_1)|^2 \geq \lambda_1^2 \int_{M^n} |\xi \cdot \psi_1|^2 = \int_{M^n} |D\psi_1|^2. \quad (3.6)$$

Hence

$$0 \leq \frac{n-3}{n-2} \int_{M^n} |\nabla^{\lambda_1} \psi_1|^2 \leq \int_{M^n} [\lambda_1^2 - \frac{n-1}{4(n-2)} R] |\psi_1|^2. \quad (3.7)$$

This immediately implies the estimate [13] given by Moroianu and Ornea. In the limiting case, $\nabla_\xi \psi_1 = \nabla_\xi^{\lambda_1} \psi_1 = 0$, moreover by (1.5)

$$\begin{aligned} \frac{1}{2} Ric(\xi) \cdot \psi_1 &= D(\nabla_\xi \psi_1) - \nabla_\xi(D\psi_1) - e_i \cdot \nabla_{\nabla_i \xi} \psi_1 \\ &= -e_i \cdot \left(\frac{\lambda_1}{n-1} \langle \nabla_i \xi, \xi \rangle \xi \cdot \psi_1 - \nabla_i \xi \cdot \psi_1 \right) \\ &= \frac{\lambda_1}{n-1} D\xi \cdot \psi_1 = 0. \end{aligned}$$

Hence, $Ric(\xi) = 0$. Furthermore, one obtains that ξ is in fact parallel by (2.4), and then we can apply Theorem 3.1 in [10] to determine the universal cover of M^n .

Remark 2. The original “twistor-like” operator $T : TM^n \otimes \Sigma M^n \rightarrow \Sigma M^n$ defined by Moroianu and Ornea is the following

$$T_X \phi \triangleq \nabla_X \phi + \frac{1}{n-1} X \cdot D\phi - \frac{1}{n-1} \langle X, \xi \rangle \xi \cdot D\phi - \langle X, \xi \rangle \nabla_\xi \phi. \quad (3.8)$$

They obtained the following important identity (see [13]).

$$\begin{aligned} \frac{n-3}{n-1} \int_{M^n} |\nabla_\xi \psi_\alpha|^2 + \int_{M^n} |T\psi_\alpha|^2 &= \int_{M^n} \left(\frac{n-2}{n-1} \lambda_\alpha^2 - \frac{1}{4} \text{Scal} \right) |\psi_\alpha|^2 \\ &\quad - \frac{1}{2(n-1)} \int_{M^n} (|D(\xi \cdot \psi_\alpha)|^2 - |D\psi_\alpha|^2), \end{aligned} \quad (3.9)$$

which holds for each eigenspinor ψ_α .

Observe that for any spinor field ϕ

$$|\nabla_\xi \phi|^2 = |-\xi \cdot \nabla_\xi \phi|^2 \leq (n-1)|T\phi|^2,$$

since

$$\sum_{i=1}^n e_i \cdot T_{e_i} \phi = -\xi \cdot \nabla_\xi \phi$$

and $T_\xi \phi = 0$. Moreover if ξ is a parallel one form of unit length, one obtains with the help of (2.6)

$$\int_{M^n} |\nabla_\xi \psi_\alpha|^2 \leq \left[\lambda_\alpha^2 - \frac{n-1}{4(n-2)} \text{Scal}_{\min} \right] \int_{M^n} |\psi_\alpha|^2, \quad (3.10)$$

which will play an important role in the following estimation.

Next, we define the operator

$$\mathcal{T}^{\lambda_N} \triangleq \nabla^{\lambda_N*} \nabla^{\lambda_N} + \frac{\text{Scal}}{4} \text{Id}, \quad (3.11)$$

where for $X \in TM^n$ and $\phi \in \Gamma(\Sigma M^n)$

$$\nabla_X^{\lambda_N} \phi \triangleq \nabla_X \phi + \frac{\lambda_N}{2(n-1)} X \cdot \phi + \frac{\lambda_N}{2(n-1)} J(X) \cdot \phi. \quad (3.12)$$

Then, it follows that

$$\begin{aligned} \int_{M^n} \langle \mathcal{T}^{\lambda_N} \phi, \phi \rangle &= \int_{M^n} \langle D^2 \phi - \frac{\lambda_N}{n-1} D\phi - \frac{\lambda_N}{n-1} D_J \phi, \phi \rangle + \frac{\lambda_N^2}{(n-1)} \int_{M^n} |\phi|^2 \\ &= \int_{M^n} |\nabla^{\lambda_N} \phi|^2 + \frac{\text{Scal}}{4} |\phi|^2. \end{aligned}$$

Next we begin to estimate the $\int_M \langle \mathcal{T}^{\lambda_N} \psi_\alpha, \psi_\alpha \rangle$ from above.

Proposition 3. *Suppose there exists a nontrivial parallel one-form on an n -dimensional closed Riemannian spin manifold (M^n, g) , $n \geq 3$ and $Scal \geq 0$, then for any positive integer $\alpha = 1, \dots, N$ we have*

$$\frac{\int_{M^n} \langle \mathcal{T}^{\lambda_N} \psi_\alpha, \psi_\alpha \rangle}{\int_{M^n} |\psi_\alpha|^2} \leq \frac{n}{n-1} \lambda_N^2 - \frac{1}{2(n-2)} Scal_{\min}. \quad (3.13)$$

Proof. Recall for any eigenspinor ψ_α such that $D\psi_\alpha = \lambda_\alpha \psi_\alpha$, $\alpha = 1, \dots, N$,

$$\begin{aligned} & \int_{M^n} \langle \mathcal{T}^{\lambda_N} \psi_\alpha, \psi_\alpha \rangle \\ &= \left(\lambda_\alpha^2 - \frac{\lambda_\alpha \lambda_N}{n-1} + \frac{\lambda_N^2}{n-1} \right) \int_{M^n} |\psi_\alpha|^2 - \frac{\lambda_N}{n-1} \int_{M^n} \langle D_J \psi_\alpha, \psi_\alpha \rangle \\ &= \left(\lambda_\alpha^2 - \frac{2\lambda_\alpha \lambda_N}{n-1} + \frac{\lambda_N^2}{n-1} \right) \int_{M^n} |\psi_\alpha|^2 + \frac{2\lambda_N}{n-1} \int_{M^n} \langle \xi \cdot \nabla_\xi \psi_\alpha, \psi_\alpha \rangle, \end{aligned}$$

where ξ is a parallel one-form of unit length. But combining the equality (2.8) with (3.10) yields

$$\lambda_\alpha \int_{M^n} \langle \xi \cdot \nabla_\xi \psi_\alpha, \psi_\alpha \rangle = \int_{M^n} |\nabla_\xi \psi_\alpha|^2 \leq \left[\lambda_\alpha^2 - \frac{n-1}{4(n-2)} Scal_{\min} \right] \int_{M^n} |\psi_\alpha|^2. \quad (3.14)$$

Hence

$$\begin{aligned} \int_{M^n} \langle \mathcal{T}^{\lambda_N} \psi_\alpha, \psi_\alpha \rangle &= \left(\lambda_\alpha^2 - \frac{2}{n-1} \lambda_\alpha \lambda_N + \frac{1}{n-1} \lambda_N^2 \right) \int_{M^n} |\psi_\alpha|^2 \\ &\quad + \frac{2\lambda_N}{n-1} \int_{M^n} \langle \xi \cdot \nabla_\xi \psi_\alpha, \psi_\alpha \rangle \\ &\stackrel{(3.14)}{\leq} \left(\lambda_\alpha^2 - \frac{2}{n-1} \lambda_\alpha \lambda_N + \frac{1}{n-1} \lambda_N^2 \right) \int_{M^n} |\psi_\alpha|^2 \\ &\quad + \frac{2\lambda_N}{(n-1)\lambda_\alpha} \left[\lambda_\alpha^2 - \frac{n-1}{4(n-2)} Scal_{\min} \right] \int_{M^n} |\psi_\alpha|^2 \\ &= \left(\lambda_\alpha^2 + \frac{1}{n-1} \lambda_N^2 \right) \int_{M^n} |\psi_\alpha|^2 \\ &\quad - \frac{\lambda_N}{2(n-2)\lambda_\alpha} Scal_{\min} \int_{M^n} |\psi_\alpha|^2. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\int_{M^n} \langle \mathcal{T}^{\lambda_N} \psi_\alpha, \psi_\alpha \rangle}{\int_{M^n} |\psi_\alpha|^2} &\leq \lambda_\alpha^2 + \frac{1}{n-1} \lambda_N^2 - \frac{\lambda_N}{2(n-2)\lambda_\alpha} Scal_{\min} \\ &\leq \frac{n}{n-1} \lambda_N^2 - \frac{1}{2(n-2)} Scal_{\min}. \end{aligned}$$

Q.E.D.

4. THE MAIN THEOREM

We are ready now to show the main theorem.

Theorem 4. *Suppose there exists a non-trivial parallel one form on an n -dimensional closed Riemannian spin manifold (M^n, g) , $n \geq 3$ and $Scal \geq 0$, then for any positive integer N , we have*

$$\lambda_N(D)^2 \geq \frac{n-1}{n} c \lambda_{k+1}(\Delta) + \frac{n-1}{4(n-2)} Scal_{\min},$$

where $k = \left\lceil \frac{N}{2^{\lfloor \frac{n}{2} \rfloor + 1}} \right\rceil$, $c = \frac{1}{8(2^{\lfloor \frac{n}{2} \rfloor + 1})^2}$.

Proof. Our strategy is to apply Theorem 2 for the modified connection ∇^{λ_N} . First, let $E_N = L(\psi_1, \dots, \psi_N)$ and F_N any N -dimensional vector subspace of $\Gamma(\Sigma M)$. Then the min-max principle gives

$$\begin{aligned} \lambda_N(\mathcal{T}^{\lambda_N}) &= \inf_{F_N} \sup_{0 \neq \varphi \in F_N} \frac{\int_{M^n} \langle \mathcal{T}^{\lambda_N} \varphi, \varphi \rangle}{\int_{M^n} |\varphi|^2} \\ &\leq \sup_{0 \neq \varphi \in E_N} \frac{\int_{M^n} \langle \mathcal{T}^{\lambda_N} \varphi, \varphi \rangle}{\int_{M^n} |\varphi|^2} \\ &= \max_{\alpha \in \{1, \dots, N\}} \frac{\int_{M^n} \langle \mathcal{T}^{\lambda_N} \psi_\alpha, \psi_\alpha \rangle}{\int_{M^n} |\psi_\alpha|^2}, \\ &\leq \frac{n}{n-1} \lambda_N^2 - \frac{1}{2(n-2)} Scal_{\min}, \end{aligned} \tag{4.1}$$

where we used Lemma 2 and the key proposition 3 above.

Secondly, the max-min principle also implies, for any positive integer i

$$\begin{aligned} \lambda_i(\Delta + \frac{Scal}{4}) &= \sup_{V_{i-1} \subset H^1(M)} \inf_{u \in V_{i-1}^\perp} \frac{\int_{M^n} (\Delta u + \frac{Scal}{4} u) u}{\int_{M^n} u^2} \\ &\geq \sup_{V_{i-1} \subset H^1(M)} \inf_{u \in V_{i-1}^\perp} \left(\frac{\int_{M^n} u \Delta u}{\int_{M^n} u^2} + \frac{Scal_{\min}}{4} \right) \\ &\geq \lambda_i(\Delta) + \frac{Scal_{\min}}{4}. \end{aligned}$$

Hence Theorem 2 implies that

$$\begin{aligned} \lambda_N(\mathcal{T}^{\lambda_N}) &\geq (1-c) \lambda_1(\Delta + \frac{1}{4} Scal) + c \lambda_{k+1}(\Delta + \frac{1}{4} Scal) \\ &\geq c \lambda_{k+1}(\Delta) + \frac{1}{4} Scal_{\min}. \end{aligned} \tag{4.2}$$

Combining the above result (4.1) with (4.2) gives

$$\lambda_N(D)^2 \geq \frac{n-1}{n} c \lambda_{k+1}(\Delta) + \frac{n-1}{4(n-2)} Scal_{\min},$$

where $k = \left\lfloor \frac{N}{2^{\lfloor \frac{n}{2} \rfloor + 1}} \right\rfloor$, $c = \frac{1}{8(2^{\lfloor \frac{n}{2} \rfloor + 1})^2}$.

Q.E.D.

5. ON LOCALLY REDUCIBLE RIEMANNIAN MANIFOLDS

Let M be a closed Riemannian spin manifold with positive scalar $Scal$. Suppose $TM = T_1 \oplus \cdots \oplus T_k$, where T_i are parallel distributions of dimension $n_i, i = 1, \dots, k$, and $n_1 > n_2 \geq \cdots \geq n_k$. Then one can define a locally decomposable Riemannian structure J as follows

$$J|_{T_1} = Id, \quad J|_{T_1^\perp} = -Id. \quad (5.1)$$

Moreover, we define the following modified metric connection and corresponding self-adjoint operator for $X \in \Gamma(TM^n)$ and $\phi \in \Gamma(\Sigma M^n)$

$$\nabla_X^{\lambda_N} \phi \triangleq \nabla_X \phi + \frac{\lambda_N}{2n_1} X \cdot \phi + \frac{\lambda_N}{2n_1} J(X) \cdot \phi \quad (5.2)$$

$$\mathcal{T}^{\lambda_N} \triangleq \nabla^{\lambda_N*} \nabla^{\lambda_N} + \frac{Scal}{4} Id. \quad (5.3)$$

Compute for any eigenspinor ψ_α such that $D\psi_\alpha = \lambda_\alpha \psi_\alpha, \alpha = 1, \dots, N$,

$$\begin{aligned} & \int_{M^n} \langle \mathcal{T}^{\lambda_N} \psi_\alpha, \psi_\alpha \rangle \\ &= \left(\lambda_\alpha^2 - \frac{2\lambda_\alpha \lambda_N}{n_1} + \frac{\lambda_N^2}{n_1} \right) \int_{M^n} |\psi_\alpha|^2 + \frac{\lambda_N}{\lambda_\alpha} \cdot \frac{1}{2n_1} \int_{M^n} |D_J \psi_\alpha - D\psi_\alpha|^2. \end{aligned}$$

But [12] gives

$$\|D\psi_\alpha\|^2 = \frac{n_1}{n_1 - 1} \|P\psi_\alpha\|^2 + \frac{1}{n_1 - 1} \sum_{i=2}^k \varepsilon_i \|D_i \psi_\alpha\|^2 + \frac{n_1}{n_1 - 1} \int_{M^n} \frac{Scal}{4} |\psi_\alpha|^2, \quad (5.4)$$

where $\varepsilon_i \triangleq \frac{n_1}{n_i} - 1 > 0$, $i = 2, \dots, k$ and D_i is the “partial” Dirac operator of subbundle T_i . Suppose

$$\{e_1, \dots, e_{n_1}, e_{n_1+1}, \dots, e_{n_1+n_2}, \dots, e_{n_1+n_2+\dots+n_{k-1}+1}, \dots, e_n\}$$

is an adapted local orthonormal frame, i.e., such that $\{e_{n_1+n_2+\dots+n_{i-1}+1}, \dots, e_{n_1+n_2+\dots+n_i}\}$ spans T_i . Then

$$D = \sum_i^k D_i \quad (5.5)$$

$$D_J = D_1 - (D_2 + \dots + D_k) \quad (5.6)$$

Therefore

$$\begin{aligned} |D_J \psi_\alpha - D \psi_\alpha|^2 &= 4 \left| \sum_{i=2}^k D_i \psi_\alpha \right|^2 \\ &\leq 4 \left(\sum_{i=2}^k |D_i \psi_\alpha| \right)^2 \\ &\leq 4 \left(\sum_{i=2}^k \varepsilon_i^{-1} \right) \sum_{i=2}^k \varepsilon_i |D_i \psi_\alpha|^2. \end{aligned}$$

Hence

$$\|D_J \psi_\alpha - D \psi_\alpha\|^2 \leq 4 \left(\sum_{i=2}^k \varepsilon_i^{-1} \right) (n_1 - 1) \left(\lambda_\alpha^2 - \frac{n_1}{4(n_1 - 1)} \text{Scal}_{\min} \right) \|\psi_\alpha\|^2,$$

which implies that

$$\begin{aligned} &\frac{\int_{M^n} \langle \mathcal{T}^{\lambda_N} \psi_\alpha, \psi_\alpha \rangle}{\int_{M^n} |\psi_\alpha|^2} \\ &\leq \left(\lambda_\alpha^2 - \frac{2\lambda_\alpha \lambda_N}{n_1} + \frac{\lambda_N^2}{n_1} \right) + \frac{\lambda_N}{\lambda_\alpha} \cdot \frac{1}{2n_1} \cdot 4 \left(\sum_{i=2}^k \varepsilon_i^{-1} \right) (n_1 - 1) \left(\lambda_\alpha^2 - \frac{n_1}{4(n_1 - 1)} \text{Scal}_{\min} \right) \\ &\leq \left(1 + 2 \sum_{i=2}^k \varepsilon_i^{-1} \right) \frac{n_1 - 1}{n_1} \lambda_N^2 - \frac{1}{2} \left(\sum_{i=2}^k \varepsilon_i^{-1} \right) \text{Scal}_{\min}. \end{aligned}$$

Theorem 5. *Let M be a compact Riemannian spin manifold with positive scalar Scal . Let $TM = T_1 \oplus \dots \oplus T_k$, where T_i are parallel distributions of dimension $n_i, i = 1, \dots, k$, and $n_1 > n_2 \geq \dots \geq n_k$. Then for any positive integer N ,*

$$\lambda_N(D)^2 \geq \frac{1}{1 + 2 \sum_{i=2}^k \varepsilon_i^{-1}} \frac{n_1}{n_1 - 1} c \lambda_{k+1}(\Delta) + \frac{n_1}{4(n_1 - 1)} \text{Scal}_{\min},$$

where $k = \left\lfloor \frac{N}{2^{\lfloor \frac{N}{2} \rfloor + 1}} \right\rfloor$, $c = \frac{1}{8(2^{\lfloor \frac{N}{2} \rfloor + 1})^2}$, $\varepsilon_i \triangleq \frac{n_1}{n_i} - 1 > 0$, $i = 2, \dots, k$.

Proof. As before, the min-max principle yields

$$\begin{aligned} c\lambda_{k+1}(\Delta) + \frac{1}{4}Scal_{\min} &\leq \lambda_N(\mathcal{T}^{\lambda_N}) \\ &\leq \left(1 + 2\sum_{i=2}^k \varepsilon_i^{-1}\right) \frac{n_1 - 1}{n_1} \lambda_N^2 - \frac{1}{2} \left(\sum_{i=2}^k \varepsilon_i^{-1}\right) Scal_{\min}, \end{aligned}$$

so we are done.

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